# ARRANGEMENT OF THE TEMPERATURE MEASUREMENT POINTS AND CONDITIONALITY OF INVERSE THERMAL CONDUCTIVITY PROBLEMS

V. V. Mikhailov

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An approach to evaluating the arrangement of temperature measurement points is presented with the aim of improving the conditionality of inverse thermal conductivity problems.

It is known that, on account of incorrect statements of inverse thermal conductivity problems (ITP), their solutions are often sensitive to errors in the initial experimental data. However, the investigation results given, for instance, in [1, 2] show that the accuracy of solutions of the boundary-value as well the coefficient ITP's depends to a considerable extent on the arrangement of the temperature measurement points. In connection with this, the problem arises of determining the conditions under which the best accuracy of ITP solutions can be secured with a limited temperature measurement accuracy.

Formally, any ITP can be stated in the form of the operator equation

$$Au = f,\ u \in U,\ f \in F, \tag{1}$$

where u is the solution to be obtained, f are the measured temperatures, and A is an operator which establishes the relationship between u and f.

For the U and F spaces, we most often use the space $L_2$ of quadratically integrable functions. The sought solution u often refers to functions appearing in the boundary conditions, boundary-value ITP's, or in the thermal conductivity equation – coefficient ITP's. The temperature is measured at certain points of the region of spatial variables under consideration. The arrangement of these points and their number are often limited for technical reasons.

The right-hand side of Eq. (1) is usually perturbed by systematic and random errors Δf, i.e., $\tilde{f} = f + \Delta f$. If the ITP is incorrect, small deviations of the right-hand side can result in considerable solution errors Δu even if the operator A is accurately assigned. We rewrite Eq. (1) with an allowance for deviations of the right-hand side and the induced deviations of the solution:

$$Au + (A + A_u)\Delta u = f + \Delta f, \tag{2}$$

where $A_u$ is the change in the operator A due to its possible nonlinearity.

The conditionality of ITP refers to the property characterizing mainly the effect of deviations of the right-hand side Δf on the solution deviation Δu. Assuming that the operator A is assigned accurately, we can introduce the conditionality measure $r = \|\Delta u\|/\|\Delta f\|$ as a qualitative characteristic of the relationship between Δf and Δu. An increase in r signifies a deterioration of the conditionality, while a decrease indicates an improvement of the conditionality.

By subtracting relationship (1) from (2), we obtain an equation relating the deviation of the ITP solution Δu to the deviation of the right-hand side Δf:

$$(A + A_u)\Delta u = \Delta f. \tag{3}$$

If the operator A is linear with respect to u, Eq. (3) becomes similar to Eq. (1).

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It is evident from (3) that the conditionality of the ITP is determined by factors affecting the properties of operator A: the dimensions of the region under investigation, the thermophysical characteristics of the material, the coordinates of the temperature measurement points, etc. Thus, for the assigned geometry of the region and the thermophysical characteristics of the material, the ITP conditionality can be regulated by choosing the coordinates of the temperature measurement points.

In order to arrive at a criterion for estimating the coordinates of the temperature measurement points, we shall consider an extremal statement of ITP. In this case, the sought solution is obtained from the condition for the minimum of the functional characterizing the deviation of the temperature T = Au, calculated for a certain value of u, from the measured temperature $\tilde{f}$, i.e., $\inf_u \|Au-\tilde{f}\|$. For the zero value of the functional, relationship (3) evidently holds.

On the other hand, this relationship makes it possible to estimate the temperature changes at the possible measurement points due to a certain Δu value by solving the direct problem $\Delta T=(A+A_u)\Delta u$. The deviation of the ITP solution ũ from the exact solution will be reduced if the coordinates X of the temperature measurement points are determined from the condition

$$\sup_X \|(A+A_u)\Delta u\|, \tag{4}$$

the realization of which for the assigned structure of operator A and the level of $\|\Delta u\|$, all other conditions being equal, leads to a reduction of r, i.e., an improvement in the conditionality of ITP.

Thus, the measurement points should be located in places where maximum temperature changes occur for a fixed variation of the solution to be determined.

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As an example, we shall consider in the region $\{0\leqslant x\leqslant b,\ t_0\leqslant t\leqslant t_k\}$ the boundary-value and the coefficient ITP's for the following thermal conductivity model:

$$C(T)\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left[\lambda(T)\frac{\partial T}{\partial x}\right],\ 0<x<b,\ t_0<t\leqslant t_k, \tag{5}$$

$$T(x,\ t_0)=T_0(x),\ 0\leqslant x\leqslant b, \tag{6}$$

$$(2-\alpha_1)T(0,t)+(1-\alpha_1)\lambda(T)\frac{\partial T(0,\ t)}{\partial x}=q_1(t), \tag{7}$$

$$(2-\alpha_2)T(b,\ t)+(1-\alpha_2)\lambda(T)\frac{\partial T(b,\ t)}{\partial x}=q_2(t), \tag{8}$$

where $\alpha_i,\ i=1,2,$ are the parameters determining the type of boundary conditions (BC) (α = 1 pertains to BC of the first kind, while α = 2 pertains to BC of the second kind). The function $T_0(x)$ is assigned.

Moreover, we assume that temperature measurements are performed at points whose coordinates are $x=X_i,\ i=\overline{1,n}$. A minimum number of points satisfies the conditions for the uniqueness of the solution of ITP.

<u>1. The Boundary-Value ITP</u>. It is necessary to determine the functions $u_1(t)=q_1(t)$ and $u_2(t)=q_2(t)$ for the general case. The dependences λ(T) and C(T) are assigned.

We assume that the functions (u)t have acquired certain small increments $\Delta u(t)$, i.e., $\tilde{u}(t)=u(t)+\Delta u(t)$. As a result of this, the temperature T (x, t) varies by a certain small amount $\vartheta(x,\ t)$, i.e., $\tilde{T}(x,t)=T(x,t)+\vartheta(x,\ t)$. Correspondingly, the thermophysical characteristics $\tilde{\lambda}(\tilde{T})=\lambda(T)+\frac{\partial\lambda}{\partial T}\vartheta$, and $\tilde{C}(\tilde{T})=C(T)+\frac{\partial C}{\partial T}\vartheta$ also change. By subtracting relationships (5)-(8) from the relationship of the perturbed problem and neglecting the products of small quantities, we obtain the expressions establishing a relationship between $\Delta u(t)$ and $\vartheta(x,\ t)$:

$$\frac{\partial\vartheta}{\partial t} = A(x,\ t)\frac{\partial^2\vartheta}{\partial x^2} + B(x,\ t)\frac{\partial\vartheta}{\partial x} + D(x,t)\vartheta, \qquad (9)$$
$$0<x<b,\ t_0<t\leqslant t_k,$$

$$\vartheta(x,\ t_0)=0,\ 0\leqslant x\leqslant b, \qquad (10)$$

$$\left[(2-\alpha_1)+(1-\alpha_1)\frac{\partial\lambda(0,\ t)}{\partial x}\right]\vartheta(0,\ t)+(1-\alpha_1)\lambda(0,\ t)\ \frac{\partial\vartheta(0,\ t)}{\partial x} = \Delta u_1(t), \qquad (11)$$

$$\left[(2-\alpha_2)+(1-\alpha_2)\frac{\partial\lambda(b,\ t)}{\partial x}\right]\vartheta(b,\ t)+(1-\alpha_2)\lambda(b,\ t)\ \frac{\partial\vartheta(b,\ t)}{\partial x} = \Delta u_2(t), \qquad (12)$$

where

$$A(x,\ t)=\lambda(x,t)/C(x,\ t);$$
$$B(x,\ t)=2\ \frac{\partial\lambda(x,\ t)}{\partial x}\Big/C(x,\ t);$$
$$D(x,\ t)=\left[\frac{\partial^2\lambda(x,\ t)}{\partial x^2}-\frac{\partial C(x,\ t)}{\partial t}\right]\Big/C(x,\ t).$$

The values of $A(x,\ t)$, $B(x,\ t)$, and D (x, t) and also $\lambda(0,\ t),\frac{\partial\lambda(0,\ t)}{\partial x}$, $\lambda(b,\ t)$, and $\frac{\partial\lambda(b,\ t)}{\partial x}$ can be calculated by solving problem (5)-(8) for the a priori assigned dependences $u_1(t)$ and $u_2(t)$.

2. The Coefficient ITP. Generally, it is necessary to determine $u_1(T)=C(T)$ and $u_2(T)=\lambda(T)$. The functions $q_1(t)$ and $q_2(t)$ are assigned.

We assume that the functions u(T) have received certain small increments, i.e., $\tilde{u}(T)=u(T)+\Delta u(T)$. As a result, the temperature T (x, t), as in the preceding problem, will change by a certain amount $\vartheta(x,\ t)$, i.e., $\tilde{T}(x,\ t)=T(x,\ t)+\vartheta(x,\ t)$.

The problem is complicated by the fact that the argument of the sought dependences is the temperature, which itself is the solution of problem (5)-(8). In order to bypass this, we shall represent the sought dependences in the form of series with respect to certain systems of basis functions,

$$u_1(T)=\sum_{j=1}^{m_1}u_{1j}\,\varphi_j(T),\ u_2(T)=\sum_{l=1}^{m_2}u_{2l}\,\psi_l(T), \qquad (13)$$

where $\varphi_j(T)$, and $\psi_l(T)$ are the basis function, $u_{ij}$ and $u_{2\ell}$ are the coefficients to be determined, and $m_1$ and $m_2$ are the numbers of terms in the corresponding series, respectively.

We write the perturbed dependences of the sought functions in the following form:

$$\tilde{u}_1(\tilde{T})=u_1(T)+k_1\,\Delta u_1(T)+\frac{\partial u_1}{\partial T}\vartheta,$$
$$\tilde{u}_2(\tilde{T})=u_1(T)+k_2\,\Delta u_2(T)+\frac{\partial u_2}{\partial T}\vartheta,$$

where k = 1 pertains to the sought dependence, while k = 0 pertains to the assigned dependence.

According to (13), $\Delta u_1=\sum_{j=1}^{m_1}\Delta u_{1j}\,\varphi_j(T),\ \Delta u_2=\sum_{l=1}^{m_2}\Delta u_{2l}\,\psi_l(T).$

As in the case of boundary-value ITP, we can write the expressions establishing a relationship between $\Delta u(T)$ and $\vartheta(x, t)$:

$$\frac{\partial \vartheta}{\partial t} = A(x, t)\frac{\partial^2 \vartheta}{\partial x^2} + B(x, t)\frac{\partial \vartheta}{\partial x} + D(x, t)\vartheta + k_2\sum_{l=1}^{m_2}\Delta u_{2l}E_l - k_1\sum_{j=1}^{m_1}\Delta u_{1j}F_j,$$

$$0 < x < b,\ t_0 < t \leqslant t_k, \tag{14}$$

$$\vartheta(x, t_0) = 0, \tag{15}$$

$$\left[(2-\alpha_1) + (1-\alpha_1)\frac{\partial u_2(0, t)}{\partial x}\right]\vartheta(0, t) + (1-\alpha_1)u_2(0, t)\frac{\partial \vartheta(0, t)}{\partial x} =$$

$$= -(1-\alpha_1)\frac{\partial T(0, t)}{\partial x}k_2\sum_{l=1}^{m_2}\Delta u_{2l}\psi_l(0, t), \tag{16}$$

$$\left[(2-\alpha_2) + (1-\alpha_2)\frac{\partial u_2(b, t)}{\partial x}\right]\vartheta(b, t) + (1-\alpha_2)u_2(b, t)\frac{\partial \vartheta(b, t)}{\partial x}$$

$$= -(1-\alpha_2)\frac{\partial T(b, t)}{\partial x}k_2\sum_{l=1}^{m_2}\Delta u_{2l}\varphi_l(b, t), \tag{17}$$

where

$$A(x, t) = u_2(x, t)/u_1(x, t);$$

$$B(x, t) = 2\frac{\partial u_2(x, t)}{\partial x}\Big/ u_1(x, t);$$

$$D(x, t) = \left[\frac{\partial^2 u_2(x, t)}{\partial x^2} - \frac{\partial u_1(x, t)}{\partial t}\right]\Big/ u_1(x, t);$$

$$E_l(x, t) = \left[\frac{\partial \psi_l(x, t)}{\partial x}\frac{\partial T(x, t)}{\partial x} + \psi_l(x, t)\frac{\partial^2 T(x, t)}{\partial x^2}\right]\Big/ u_1(x, t),\ l = \overline{1, m_2};$$

$$F_j(x, t) = \varphi_j(x, t)\frac{\partial T(x, t)}{\partial t}\Big/ u_1(x, t),\ j = \overline{1, m_1}.$$

The dependences $A(x, t)$, $B(x, t)$, $D(x, t)$, $E_l(x, t)$, and $F_j(x, t)$ and also

$$u_2(0, t),\ \frac{\partial u_2(0, t)}{\partial x},\ \frac{\partial T(0, t)}{\partial x},\ \psi_l(0, t),\ u_2(b, t),\ \frac{\partial u_2(b, t)}{\partial x},$$

$$\frac{\partial T(b, t)}{\partial x},\ \psi_l(b, t),$$

can be calculated, as in the preceding problem, by solving problem (5)-(8) for the a priori assigned dependences $u_1(T)$ and $u_2(T)$

In solving the extremal ITP, the unknown functions are usually determined from the condition for the minimum of the root-mean-square discrepancy,

$$J(u) = \sum_{i=1}^{n}\sigma_i\int_{t_0}^{t_k}[T(X_i, t) - f_i(t)]^2 dt, \tag{18}$$

where n is the number of measurement points, $X_i$, $i = \overline{1, n}$, are the coordinates of the temperature measurement points, $f_i(t)$ is the temperature measured at the i-th point, and $\sigma_i$, $i = \overline{1, n}$, are the weighting factors.

For estimating the change in functional (18) caused by small deviations of the sought functions Δu, the following relationship holds:

$$\Delta J(\Delta u) = \sum_{i=1}^{n} \sigma_i \int_{t_0}^{t_k} \{2[T(X_i, t) - f_i(t)]\vartheta(X_i, t) + \vartheta^2(X_i, t)\} dt, \quad (19)$$

where $\vartheta(X_i, t)$, $i=\overline{1, n}$, is the solution of problem (9)-(12) or (14)-(17) at the measurement points.

In the neighborbood of the sought solution of the ITP, i.e., for $T(X_i, t) \simeq f_i(t)$, $i=\overline{1, n}$, the value of $\Delta J(\Delta u)$ can be estimated as

$$\Delta \bar{J}(\Delta u) = \sum_{i=1}^{n} \sigma_i \int_{t_0}^{t_k} \vartheta^2(X_i, t) dt. \quad (20)$$

Evidently, the greater the effect of the sourght functions on the temperature at the measurement points $T(X_i, t)$ the larger the value of functional (20) for a certain fixed Δu value, and the coordinates of the measurement points must satisfy the condition $\sup_X \Delta \bar{J}(\Delta u)$ (analog of condition (4) in the splace $L_2$).

On the other hand, for a fixed level of the measurement error $\|\Delta f_i\|$, $i=\overline{1, n}$, fulfillment of this condition ensures a minimum deviation of the sought ITP solution from the exact one.

A quantitative estimate of the terms in relationship (20) corresponding to different coordinates $X_i$, $i=\overline{1, n}$, can be obtained by solving the above boundary-value problem (9)-(12) or (14)-(17) for the temperature deviations $\vartheta(x, t)$.

In the case of a boundary-value ITP in linear formulation, i.e.,for $C(T) = C_0$ = const and $\lambda(T) = \lambda_0$ = const, relationships (5)-(8), as was mentioned above. In this case, functional (20) will evidently have a maximum value, while the conditionality of the problem will be at its best if the measurement points are arranged in the vicinity of the boundaries of the region under investigation, where it is necessary to determine the relationship $u_1(t)$ or $u_2(t)$.

This is supported, in particular, by the results given in [1].

In the case of the coefficient ITP, the results are not that obvious even if $u_1(T) = u_{11}$=const, and $u_2(T) = u_{21}$=const, since Eq. (14) differs from Eq. (5) by the presence of terms proportional to $\partial^2 T/\partial x^2$ and $\partial T/\partial t$, respectively.

## NOTATION

$T(x, t)$, temperature; $C(T)$, volumetric heat; $\lambda(T)$, thermal conductivity coefficient; $T_0(x)$, initial temperature distribution; t, time; x, space coordinate; $t_0$ and $t_k$, beginning and end of the time interval, respectively; b, end of the space variable interval; $q_1(t)$ and $q_2(t)$, boundary functions.

## LITERATURE CITED


1. O. M. Alifanov and V. V. Mikhailov, Teplofiz. Vys. Temp., 21, No. 5, 944-951 (1983).
2. E. A. Artyukhin and A. S. Okhapkin, Inzh. Zh., 45, No. 5, 781-788 (1983).